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A note on the existence of global solutions for reaction-diffusion equations with almost-monotonic nonlinearities

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1 Introduction

Let us consider the following problem

$$\begin{cases} u_t - \Delta u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \\ u(0) = u_0 & \text{in } \Omega \end{cases} \quad (1.1)$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, with f of the form

$$f(x, u) = g(x) + m(x)u + f_0(x, u), \quad x \in \Omega, \quad u \in \mathbb{R}, \quad (1.2)$$

where

$$g \in L^{q_0}(\Omega), \text{ with } N/2 < q_0 < \infty, \quad m \in L^{r_0}(\Omega) \text{ with } N/2 < r_0 \leq \infty \quad (1.3)$$

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and f_0 is a Carathéodory function, locally Lipschitz in the second variable with an $L^{q_0}(\Omega)$ Lipschitz constant, i.e, such that for $|u| \leq R$ and $|v| \leq R$ we have

$$|f_0(x, u) - f_0(x, v)| \leq L_0(x, R)|u - v| \quad (1.4)$$

with $0 \leq L_0(\cdot, R) \in L^{q_0}(\Omega)$ for each $R > 0$, and with $f_0(\cdot, 0) = \partial_u f_0(\cdot, 0) = 0$.

Our goal here is to prove global existence and uniqueness of solutions for initial data $u_0 \in L^q(\Omega)$ for any $1 \leq q < \infty$ under the only additional assumption that $f_0(x, u)$ is *almost monotonic*, i.e, to satisfy the following condition

$$\partial_u f_0(x, u) \leq L(x), \quad \text{for all } x \in \Omega, u \in \mathbb{R} \quad (1.5)$$

for some $L \in L^{\sigma_0}(\Omega)$, $\sigma_0 > N/2$. Note that this implies

$$uf(x, u) \leq C(x)u^2 + D(x)|u|, \quad x \in \Omega, u \in \mathbb{R}, \quad (1.6)$$

for $C = m + L \in L^\sigma(\Omega)$, $\sigma = \min\{r_0, \sigma_0\} > N/2$ and $0 \leq D = |g| \in L^{q_0}(\Omega)$.

This paper is organised as follows. In Section 2 we recall some known results about local and global existence of solutions for problem (1.1) with smooth enough initial data. In Section 3, we proof our main results. Namely, problem (1.1) is well-posed in $L^q(\Omega)$ for every $1 \leq q < \infty$ and its solutions are unique and globally defined, see Theorems 3.1 and 3.3. Finally, in Section 4 we make some remarks on the asymptotic behaviour of the solutions of the problems, compare the results with known ones, and give some examples.

2 Preliminary results

Notice that, from (1.4), the Nemytskii operator associated to $F(x, u) = g(x) + f_0(x, u)$ is Lipschitz on bounded sets of $L^\infty(\Omega)$ into of $L^{q_0}(\Omega)$ and so we have the existence of local solutions of problem (1.1) for smooth initial data. Namely, using the standard Bessel–Lebesgue spaces $H^{\alpha, q}(\Omega)$, we have using the techniques in [5]

Theorem 2.1 *Assume that f is as in (1.2)–(1.3). If $\alpha < 1$ is such that $2\alpha - \frac{N}{q_0} > 0$, then for any initial data $u_0 \in H^{2\alpha, q_0}(\Omega) \cap H_0^{1, q_0}(\Omega)$ there exists a unique local solution of (1.1) with initial data u_0 , with $u(\cdot; u_0) \in C([0, \tau]; H^{2\alpha, q_0}(\Omega) \cap H_0^{1, q_0}(\Omega)) \cap C((0, \tau); H^{2, q_0}(\Omega))$ and $u_t \in C((0, \tau); H^{2\gamma, q_0}(\Omega) \cap H_0^{1, q_0}(\Omega))$ for any $\gamma < 1$, for some τ depending on u_0 .*

The solution satisfies the Variation of Constant Formula,

$$u(t; u_0) = S_m(t)u_0 + \int_0^t S_m(t-s)(g + f_0(u(s; u_0))) \, ds, \quad t > 0,$$

where S_m denotes the semigroup generated by $\Delta + m(x)I$ with Dirichlet boundary conditions.

Note that since $q_0 > N/2$, $H^{2\alpha, q_0}(\Omega) \cap H_0^{1, q_0}(\Omega) \subset C_0^\eta(\overline{\Omega})$ for any $0 \leq \eta < 2 - N/q_0$. In particular the solution satisfies $u \in C([0, \tau] \times \overline{\Omega})$. Also, if g and $L_0(\cdot, R)$, for each R , are a bounded functions, then the above holds for any $N/2 < q_0 < \infty$.

On the other hand, it is known that condition (1.6) ensures the global existence of the local solutions in Theorem 2.1 (see [1] and [9]) and so we have $u \in C([0, \infty) \times \overline{\Omega})$. Namely, we have

Theorem 2.2 Assume that f is as in (1.2)–(1.3) and satisfies (1.6), that is

$$uf(x, u) \leq C(x)u^2 + D(x)|u|, \quad x \in \Omega, \quad u \in \mathbb{R},$$

for some $C \in L^\sigma(\Omega)$ and $0 \leq D \in L^\rho(\Omega)$ with $\sigma, \rho > N/2$.

Then for the solutions in Theorem 2.1 one has $\tau = \infty$.

3 Existence in $L^q(\Omega)$, $1 \leq q < \infty$

The goal of the section is to prove main result in this work. Namely, the existence of a unique solution of problem (1.1) starting at $u_0 \in L^q(\Omega)$ for any $1 \leq q < \infty$. We start now with the existence part.

Theorem 3.1 Let $1 \leq q < \infty$. Suppose that f is as in (1.2)–(1.3) and f_0 satisfies (1.5) with $L \in L^{\sigma_0}(\Omega)$, $\sigma_0 > N/2$. Then, for any $u_0 \in L^q(\Omega)$, there exists a solution of (1.1) defined for all $t \geq 0$, u , such that

$$u \in C([0, \infty); L^q(\Omega)) \cap C((0, \infty); H^{2,q_0}(\Omega) \cap H_0^{1,q_0}(\Omega)), \quad u(0) = u_0$$

and satisfies

$$u(t) = S_m(t)u_0 + \int_0^t S_m(t-s)(g + f_0(\cdot, u(s))) ds \quad (3.1)$$

where S_m denotes the semigroup generated by $\Delta + m(x)I$ with Dirichlet boundary conditions.

Moreover, for each $T > 0$ there exists $c(T)$ such that

$$|u(t, x)| \leq c(T) \left(1 + t^{-\frac{N}{2q}} \|u_0\|_{L^q(\Omega)}\right), \quad 0 < t \leq T \quad \text{for all } x \in \bar{\Omega}. \quad (3.2)$$

Proof. We proceed in several steps. In the first step, fixed $1 \leq q < \infty$, we construct a Cauchy sequence of approximating solutions. Then, we obtain a uniform $L^\infty(\Omega)$ bound for the approximating sequence. In a third step, we show that the limit of the approximating solutions is a solution of the limit problem (notice that such limit exists since the approximating solutions forms a Cauchy sequence). Finally, we show how to obtain more regularity of the solution constructed in the previous steps.

Step 1. Approximate the initial data. Let $\alpha < 1$ such that $2\alpha - \frac{N}{q_0} > 0$. Then, by Theorem 2.1, the problem (1.1) is well-posed in $H^{2\alpha, q_0}(\Omega) \cap H_0^{1, q_0}(\Omega)$. Also, since f satisfies (1.6), the solutions are globally defined for $t > 0$, see Theorem 2.2 and (1.6).

Hence, for any $1 \leq q < \infty$ and $u_0 \in L^q(\Omega)$, we can take smooth enough initial data $u_0^n \in H^{2\alpha, q_0}(\Omega) \cap H_0^{1, q_0}(\Omega)$ such that $u_0^n \rightarrow u_0$ in $L^q(\Omega)$ as $n \rightarrow \infty$ and consider the solutions of (1.1) starting at u_0^n . We define $u_n(t) = u(t; u_0^n)$.

Let $v_{n,k}(t) = u_n(t) - u_k(t)$. Subtracting equations for u_n and u_k , we have

$$\begin{cases} \partial_t v_{n,k}(t) - \Delta v_{n,k}(t) = m(x)v_{n,k}(t) + f_0(x, u_n(t)) - f_0(x, u_k(t)) & \text{in } \Omega \\ v_{n,k} = 0 & \text{on } \Gamma \\ v_{n,k}(0) = u_n^0 - u_k^0 & \text{in } \Omega. \end{cases}$$

Observe that given $T > 0$, for fixed n, k , we have for almost all $x \in \Omega, t \in [0, T]$ and some $0 < \theta(t, x) < 1$

$$\begin{aligned} f_0(x, u_n(t, x)) - f_0(x, u_k(t, x)) &= \partial_u f_0(x, \theta(t, x)u_n(t, x) + (1 - \theta(t, x))u_k(t, x))v_{n,k}(t, x) \\ &= C_{n,k}(t, x)v_{n,k}(t, x) \end{aligned}$$

for some function $C_{n,k}(t, x)$. Notice that $C_{n,k} \in L^\infty((0, T); L^{q_0}(\Omega))$ since u_n and u_k are smooth, f_0 is locally Lipschitz in the second variable with an $L^{q_0}(\Omega)$ Lipschitz constant. Also, from (1.5) we have $C_{n,k}(x, t) \leq L(x)$ for all $t \geq 0$ and $x \in \Omega$.

Now, consider the linear problem

$$\begin{cases} z_t - \Delta z = (m(x) + C_{n,k}(t, x))z & \text{in } \Omega \\ z = 0 & \text{on } \Gamma \\ z(0) = z_0 & \text{in } \Omega \end{cases} \quad (3.3)$$

with z_0 smooth and denote by $z(t, 0; z_0)$ the solution, whose existence follows from [7] or [3]. Such solutions satisfy, by comparison, $z(t, 0; -|z_0|) \leq z(t, 0; z_0) \leq z(t, 0; |z_0|)$, i.e.,

$$|z(t, 0; z_0)| \leq z(t, 0; |z_0|), \quad t \geq 0,$$

and the latter is a nonnegative solution of (3.3).

But for nonnegative initial data, $z_0 \geq 0$, since $C_{n,k}(x, t) \leq L(x)$ for all $t \geq 0$ and $x \in \Omega$, we can compare $z(t, 0; z_0) \geq 0$ with the solutions of

$$\begin{cases} w_t - \Delta w = (m(x) + L(x))w & \text{in } \Omega \\ w = 0 & \text{on } \Gamma \\ w(0) = z_0 \geq 0 & \text{in } \Omega \end{cases}$$

to obtain $0 \leq z(t, 0; z_0) \leq w(t; z_0)$ for any $t \geq 0$.

Hence, we obtain that for any smooth initial data z_0 in (3.3) we have

$$|z(t, 0; z_0)| \leq w(t; |z_0|) \quad \text{for } t \geq 0.$$

In particular,

$$\|z(t, 0; z_0)\|_{L^q(\Omega)} \leq \|w(t; |z_0|)\|_{L^q(\Omega)} \leq ce^{-\lambda t} \|z_0\|_{L^q(\Omega)}$$

where λ is the first eigenvalue of $-\Delta - (m(x) + L(x))I$ on Ω with Dirichlet boundary conditions.

Now, $v_{n,k}$ is a solution of (3.3) and so

$$\|v_{n,k}(t)\|_{L^q(\Omega)} \leq ce^{-\lambda t} \|v_{n,k}(0)\|_{L^q(\Omega)}$$

for all $t \geq 0$. In particular, given $T > 0$, we have that for any $0 \leq t \leq T$,

$$\|v_{n,k}(t)\|_{L^q(\Omega)} \leq c(T) \|v_{n,k}(0)\|_{L^q(\Omega)} \rightarrow 0 \quad \text{as } n, k \rightarrow \infty$$

and so, u_n is a Cauchy sequence in $C([0, T]; L^q(\Omega))$.

Hence, there exists $u \in C([0, \infty); L^q(\Omega))$ such that for any $T > 0$,

$$\sup_{t \in [0, T]} \|u_n(t) - u(t)\|_{L^q(\Omega)} \leq c(T) \|u_n^0 - u_0\|_{L^q(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3.4)$$

i.e, for any $T > 0$,

$$u_n \rightarrow u \quad \text{in } C([0, T]; L^q(\Omega)).$$

In particular, passing to a subsequence if needed, we can assume $u_n(t, x) \rightarrow u(t, x)$ as $n \rightarrow \infty$ a.e. for $(t, x) \in [0, T] \times \Omega$.

Also it is easy to see that u does not depend on the sequence of initial data, but only on $u_0 \in L^q(\Omega)$.

Step 2. L^∞ -bound for the approximating sequence. Let us show now that the sequence $u_n(t)$ is uniformly bounded in $L^\infty(\Omega)$ with respect to n , for $0 < \varepsilon \leq t \leq T$.

For this, since f satisfies (1.6), we will use the auxiliary problem

$$\begin{cases} U_t - \Delta U = C(x)U + D(x) & \text{in } \Omega \\ U = 0 & \text{on } \Gamma \\ U(0) & \text{given in } L^q(\Omega) \end{cases} \quad (3.5)$$

with $C = m + L \in L^\sigma(\Omega)$, $\sigma = \min\{r_0, \sigma_0\} > N/2$ and $0 \leq D = |g| \in L^{q_0}(\Omega)$, $q_0 > N/2$.

Denote by $U^n(t, x)$ the solution of (3.5) with initial data $|u_0^n|$ and by $U(t, x)$ the solution of (3.5) with initial data $|u_0|$.

Now, using the variation of constants formula in (3.5) we have

$$U^n(t) = \Phi(t) + U_h^n(t), \quad U(t) = \Phi(t) + U_h(t)$$

where $U_h^n(t), U_h(t)$ are the solutions of the homogeneous problem

$$\begin{cases} V_t - \Delta V = C(x)V & \text{in } \Omega \\ V = 0 & \text{on } \Gamma \\ V(0) & \text{given in } L^q(\Omega) \end{cases}$$

resulting from taking $D \equiv 0$ in (3.5) and initial data $|u_0^n|$ and $|u_0|$ respectively, and $\Phi(t)$ is the unique solution of problem (3.5) with $U(0) = 0$ (which does not depend on u_0^n or u_0), that is,

$$\begin{cases} W_t - \Delta W = C(x)W + D(x) & \text{in } \Omega \\ W = 0 & \text{on } \Gamma \\ W(0) = 0 & \text{in } \Omega. \end{cases}$$

In other words $U_h^n(t) = S_C(t)|u_0^n|$, $U_h(t) = S_C(t)|u_0|$ and $\Phi(t) = \int_0^t S_C(t-s)D \, ds$ where S_C denotes the semigroup generated by $\Delta + C(x)I$ with Dirichlet boundary conditions. Hence standard linear parabolic estimates implies that, for any $T > 0$,

$$\|U^n(t)\|_{L^\infty(\Omega)} \leq c(T)(1 + t^{-\frac{N}{2q}} \|u_0^n\|_{L^q(\Omega)}), \quad 0 < t \leq T,$$

$$\|U^n(t)\|_{L^q(\Omega)} \leq c(T)(1 + \|u_0^n\|_{L^q(\Omega)}), \quad 0 \leq t \leq T.$$

Also,

$$\begin{aligned} \|U^n(t) - U(t)\|_{L^\infty(\Omega)} &= \|U_h^n(t) - U_h(t)\|_{L^\infty(\Omega)} = \|S_C(t)(|u_0^n| - |u_0|)\|_{L^\infty(\Omega)} \\ &\leq c(T)t^{-\frac{N}{2q}} \| |u_0^n| - |u_0| \|_{L^q(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

for $0 < t \leq T$, and for $0 \leq t \leq T$

$$\|U^n(t) - U(t)\|_{L^q(\Omega)} \leq c(T) \| |u_0^n| - |u_0| \|_{L^q(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, for any $0 < \varepsilon < T < \infty$,

$$U^n \rightarrow U \quad \text{in } L^\infty([\varepsilon, T] \times \Omega) \cap C([0, T]; L^q(\Omega)).$$

Observe now that, since f satisfies (1.6), $U^n(t, x)$ is a supersolution of problem (1.1) and $-U^n(t, x)$ is a subsolution. Thus,

$$|u_n(t, x)| \leq U^n(t, x) \leq c(T)(1 + t^{-\frac{N}{2q}} \|u_0^n\|_{L^q(\Omega)}), \quad 0 < t \leq T, \quad \text{a.e. in } \Omega, \quad (3.6)$$

and so

$$\|u_n(t)\|_{L^\infty(\Omega)} \leq c(\varepsilon, T, \|u_0^n\|_{L^q(\Omega)}), \quad \varepsilon \leq t \leq T.$$

Now, since $u_0^n \rightarrow u_0$ in $L^q(\Omega)$ as $n \rightarrow \infty$ and the convergences $U^n(t, x) \rightarrow U(t, x)$ and $u_n(t, x) \rightarrow u(t, x)$ obtained above (see (3.4)) we get

$$|u(t, x)| \leq U(t, x) \leq c(T)(1 + t^{-\frac{N}{2q}} \|u_0\|_{L^q(\Omega)}), \quad 0 < t \leq T, \quad \text{for a.e. } x \in \Omega. \quad (3.7)$$

Now observe that the bounds above, the regularity of u_n in Theorem 2.1 and (3.4) imply that for any $0 < \varepsilon < T < \infty$ and $1 \leq s < \infty$,

$$\sup_{t \in [\varepsilon, T]} \|u_n(t) - u(t)\|_{L^s(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3.8)$$

i.e, for any $T > 0$ and $1 \leq s < \infty$,

$$u_n \rightarrow u \quad \text{in } C([\varepsilon, T]; L^s(\Omega)).$$

In particular $u \in C((0, \infty); L^s(\Omega))$ for any $1 \leq s < \infty$.

Step 3. The limit is a solution of (1.1). First, assume $0 < \varepsilon < t < T$. Then for any $\phi \in H^{2, q'_0}(\Omega) \cap H_0^{1, q'_0}(\Omega)$, where q'_0 is the conjugate of q_0 , i.e, $\frac{1}{q_0} + \frac{1}{q'_0} = 1$ (as usual, for $q_0 = 1$ we take $q'_0 = \infty$), we have from (1.1) and the regularity of u_n in Theorem 2.1,

$$\frac{d}{dt} \int_{\Omega} u_n \phi + \int_{\Omega} u_n (-\Delta \phi) = \int_{\Omega} f(\cdot, u_n) \phi = \int_{\Omega} g \phi + \int_{\Omega} m(x) u_n \phi + \int_{\Omega} f_0(x, u_n) \phi.$$

Now, using the uniform bounds in (3.6), (3.7) and the convergence in (3.8), and the fact that f_0 is locally Lipschitz in its second variable with an $L^{q_0}(\Omega)$ Lipschitz constant, we have that for $1 \leq s \leq q_0$,

$$f_0(\cdot, u_n) \rightarrow f_0(\cdot, u) \quad \text{in } C([\varepsilon, T]; L^s(\Omega)).$$

Hence, letting $n \rightarrow \infty$, we get

$$\frac{d}{dt} \int_{\Omega} u \phi + \int_{\Omega} u (-\Delta \phi) = \int_{\Omega} f(\cdot, u) \phi = \int_{\Omega} g \phi + \int_{\Omega} m(x) u \phi + \int_{\Omega} f_0(x, u) \phi.$$

Notice that from [2], this implies

$$u(t) = S_m(t - \varepsilon)u(\varepsilon) + \int_{\varepsilon}^t S_m(t - s)h(s) ds \quad (3.9)$$

where $S_m(t)$ denotes the strongly continuous analytic semigroup generated by $\Delta + m(x)I$ with homogeneous Dirichlet boundary conditions, and $h(\cdot) = g + f_0(\cdot, u(\cdot)) \in L^\infty([\varepsilon, T]; L^{q_0}(\Omega))$.

The smoothing effect of the semigroup gives that

$$\int_{\varepsilon}^t S_m(t - s)h(s) ds \in C([\varepsilon, T]; H^{2\gamma, q_0}(\Omega) \cap H_0^{1, q_0}(\Omega)), \text{ for any } \gamma < 1,$$

while the continuity of the linear semigroup $S_m(t)$ at 0 and $u(\varepsilon) \rightarrow u_0$ in $L^q(\Omega)$ as $\varepsilon \rightarrow 0$, give, taking $\varepsilon \rightarrow 0$ in (3.9),

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^t S_m(t - s)h(s) ds = u(t) - S_m(t)u_0.$$

Thus,

$$u(t) = S_m(t)u_0 + \int_0^t S_m(t - s)(g + f_0(s, u(s))) ds.$$

Step 4. Further regularity. From the smoothing effect of the semigroup $S_m(t)$ and the regularity observed above, we have that for any $\varepsilon > 0$, $u(\varepsilon) \in H^{2\alpha, q_0}(\Omega) \cap H_0^{1, q_0}(\Omega)$ for some $\alpha < 1$ such that $2\alpha - \frac{N}{q_0} > 0$.

Therefore, for $t \geq \varepsilon$, $u(t)$ coincides with the unique solution in Theorems 2.1 and 2.2. In particular $u(t)$ is continuous in Ω and we can take $x \in \bar{\Omega}$ in (3.7). ■

Now, from (3.7) and standard L^p - L^q estimates for (3.5) we have,

Corollary 3.2 *For $1 \leq q < \infty$ and $T > 0$, we have that the solution u in Theorem 3.1 satisfies, for $q \leq p \leq \infty$,*

$$\|u(t)\|_{L^p(\Omega)} \leq c(T) \left(1 + t^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p})}\right) \|u_0\|_{L^q(\Omega)} \quad 0 < t \leq T.$$

Let show now that the solutions of (1.1) in $L^q(\Omega)$ are unique for $1 \leq q < \infty$.

Theorem 3.3 *Given $u_0 \in L^q(\Omega)$, $1 \leq q < \infty$, there exists a unique function*

$$v \in C([0, \infty); L^q(\Omega)) \cap L_{\text{loc}}^\infty((0, \infty); L^\infty(\Omega)), \quad v(0) = u_0$$

that satisfies

$$v(t) = S_m(t)u_0 + \int_0^t S_m(t - s)(g + f_0(\cdot, v(s))) ds, \quad t \geq 0 \quad (3.10)$$

where S_m denotes the semigroup generated by $\Delta + m(x)I$ with Dirichlet boundary conditions.

Therefore the function $u(\cdot)$ constructed in Theorem 3.1 is the unique function satisfying this.

Proof. Notice that the function u constructed in Theorem 3.1 satisfies the assumptions above. So, let v also satisfy the statement of the theorem. Then from (3.10) we have that, for any $\varepsilon > 0$,

$$v(t) = S_m(t - \varepsilon)v(\varepsilon) + \int_{\varepsilon}^t S_m(t - s)(g + f_0(\cdot, v(s))) ds.$$

From the assumptions on v we have that $h(s) = g + f_0(\cdot, v(s))$ satisfies, for any $T > 0$, that $h \in L^\infty([\varepsilon, T]; L^{q_0}(\Omega))$. Then, the smoothing effect of the semigroup gives that

$$v \in C([\varepsilon, T]; H^{2\gamma, q_0}(\Omega) \cap H_0^{1, q_0}(\Omega)), \text{ for any } \gamma < 1.$$

Hence, for $t \geq \varepsilon$, v is a solution as in Theorem 2.1.

Hence, arguing as in (3.8) we have

$$\sup_{\varepsilon \leq t \leq T} \|u(t) - v(t)\|_{L^q(\Omega)} \leq c(T)\|u(\varepsilon) - v(\varepsilon)\|_{L^q(\Omega)}$$

with $c(T)$ not depending on ε .

The continuity of u and v at 0 in $L^q(\Omega)$, and the fact that $u(0) = v(0)$ imply $u = v$.

■

4 Final remarks and examples

(i) Note that Theorems 3.1 and 3.3 allow to define a strongly continuous nonlinear semigroup in $L^q(\Omega)$ as

$$S(t)u_0 = u(t; u_0), \quad t \geq 0$$

where $u(t; u_0)$ is the solution in Theorem 3.1.

The asymptotic behavior of this semigroup is the same as the semigroup obtained for more regular initial data from Theorems 2.1 and 2.2. In fact, from (3.2) we get that for any $0 < \varepsilon < T < \infty$ and for any bounded set of initial data $B \subset L^q(\Omega)$ we get that

$$\{S(t)B, \varepsilon \leq t \leq T\} \text{ is bounded in } L^\infty(\Omega).$$

This implies, in turn that

$$\{g + f_0(\cdot, u(t; u_0)), \varepsilon \leq t \leq T, u_0 \in B\} \text{ is bounded in } L^{q_0}(\Omega)$$

and again the smoothing effect of the semigroup implies that

$$\{S(t)B, \varepsilon \leq t \leq T\} \text{ is bounded in } H^{2\gamma, q_0}(\Omega) \cap H_0^{1, q_0}(\Omega)$$

for any $\gamma < 1$.

In particular, the compact attractor found in [1] and [9] also attracts the solutions of Theorem 3.1, starting in bounded sets of $L^q(\Omega)$ for any $1 \leq q < \infty$.

(ii) Notice that similar results as here have been obtained in [1], [4] and [8]. However those proof are based on energy estimates of the approximating solutions, while the proof presented above is based on the maximum principle, in the form of the comparison principle.

In particular, for the case of initial data in $L^1(\Omega)$, using the former technique would require using Kato's inequality which turns out to be not "dissipative enough" to obtain Theorem 3.1. In fact in that case one would require $L \in L^\infty(\Omega)$; see [4]. So, our proof provides a unified argument for the cases $q = 1$ and $1 < q < \infty$.

(iii) The standard theory for semilinear reaction-diffusion equations requires f to satisfy some growth restriction in order to obtain a local well-posed problem in $L^q(\Omega)$. Namely, the equation (1.1) is locally well posed provided f satisfies

$$|f(x, t) - f(x, s)| \leq C(1 + |s|^{p-1} + |s|^{p-1})|t - s|, \quad t, s \in \mathbb{R} \quad (4.11)$$

for all $x \in \Omega$, with

$$p \leq p_c = 1 + \frac{2q}{N} \quad \left(\text{i.e, for any } q \geq q_c = \frac{N(p-1)}{2} \right).$$

Our results in this paper use no growth assumption whatsoever.

(iv) Theorems 3.1 and 3.3 extend to problems in unbounded domains in a natural way (see [1]). Also, the same techniques can be applied to obtain solutions in \mathbb{R}^N in any locally uniform space, $L^q_U(\Omega)$, see [4] for a proof based on energy estimates. In the case of initial data in $L^1_U(\Omega)$, L in (1.5) was required in [4] to be bounded. By the techniques presented here, no such additional restriction is required on L in order to obtain a solution.

(v) In [6], positive solutions of equation $u_t - \Delta u = -|u|^p$ with measures as initial data is considered. In particular, for positive $L^1(\Omega)$ densities, the solution is unique. We have shown that this uniqueness also holds for general $L^1(\Omega)$ initial data (with no assumption on their sign).

(vi) An example of nonlinearity for which all the previous results apply is the following:

$$f_0(x, u) = \sum_{j=1}^k n_j(x) h_j(u) + f_1(x, u)$$

with $h_j \in C^1(\mathbb{R})$, $h_j(0) = h'_j(0) = 0$, $j = 1, \dots, k$, and $f_1(x, u)$ is a Hölder continuous with respect to x uniformly for u in bounded sets of \mathbb{R} , $\partial_s f_1(x, u)$ is bounded in x for u in bounded sets of \mathbb{R} and $f_1(x, 0) = \partial_u f_1(x, 0) = 0$, $x \in \Omega$. This includes in particular the following cases, taking $f_1 \equiv 0$:

- *Logistic equation*

$$f_0(x, u) = -n(x)|u|^{\rho-1}u$$

with $n(x)$ a nonnegative $L^d(\Omega)$ function, with $d > N/2$, not identically zero, and $\rho > 1$. In this case, $L_0(x, R) = \rho R^{\rho-1}n(x) \in L^r(\Omega)$ and so $q_0 = d$. Also, we can take $L \equiv 0$ in (1.5).

- Polynomial nonlinearity

$$f_0(x, u) = \sum_{j=2}^k n_j(x) u^j$$

with k odd, and $n_j \in L^{d_j}(\Omega)$, with $d_j > N/2$ to be made precise below, for $2 \leq j \leq k$, and $n_k(x) \leq a_0 < 0$ for all $x \in \Omega$. In this case, we can take

$$L_0(x, R) = \sum_{j=2}^k (j-1) R^{j-1} |n_j(x)|,$$

thus $L_0 \in L^{q_0}(\Omega)$ with $q_0 = \min\{d_1, \dots, d_k\} > N/2$.

Also, $\partial_s f_0(x, u)$ is bounded above by

$$\max_{u \in \mathbb{R}} \left[k n_k(x) u^{k-1} + \sum_{j=2}^{k-1} j n_j(x) u^{j-1} \right]. \quad (4.12)$$

Since $k-1$ is even, using Young's inequality each term $|j n_j(x) u^{j-1}|$, for $2 \leq j \leq k-1$, is bounded above by $\varepsilon u^{k-1} + C(\varepsilon) |n_j(x)|^{(k-1)/(k-j)}$ for any $\varepsilon > 0$.

Also, since $n_k(x) < a_0 < 0$, for sufficiently small $\varepsilon > 0$, (4.12) is bounded above by

$$\frac{k a_0}{2} u^{k-1} + C(\varepsilon) \sum_{j=2}^{k-1} |n_j(x)|^{(k-1)/(k-j)},$$

and we can then take,

$$L(x) = C(\varepsilon) \sum_{j=2}^{k-1} |n_j(x)|^{(k-1)/(k-j)}.$$

Then, $L \in L^{\sigma_0}(\Omega)$, with

$$\sigma_0 = \min_{j=2, \dots, k-1} \left\{ \frac{d_j(k-j)}{(k-1)} \right\} < q_0,$$

which we need to be larger than $N/2$, that is, $d_j > N \frac{(k-1)}{2(k-j)}$, $j = 2, \dots, k-1$.

- Polynomial nonlinearity with monotone powers

$$f_0(x, u) = \sum_{j=1}^k n_j(x) |u|^{\rho_j-1} u$$

with $n_k(x) \leq a_0 < 0$, $x \in \Omega$ and $1 < \rho_j < \rho_k$, $n_j \in L^{d_j}(\Omega)$, $d_j > N/2$ for $1 \leq j \leq k$. We can take L and L_0 analogous to the previous example.

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